A GROBMAN–HARTMAN THEOREM FOR A DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT GENERALIZED ARGUMENT

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Abstract. We obtain sufficient conditions ensuring the existence of a uniformly continuous and Hölder continuous homeomorphism between the solutions of a linear system of differential equations with piecewise constant argument of generalized type and the solutions of the quasilinear corresponding system. We use a definition (recently introduced by M. Akhmet) of exponential dichotomy for those systems combined with technical assumptions on the nonlinear part. Our result generalizes a previous work of G. Papaschinopoulos.

1. Introduction

The purpose of this article is to study the strong topological equivalence (see *e.g.*, [15, 16, 22, 31] for definitions) between the solutions of the linear differential equation with piecewise constant arguments of generalized type:

$$\dot{y}(t) \quad = \quad A(t)y(t) + A_0(t)y(\gamma(t)), \tag{1.1}$$

and the family of nonlinear systems

$$\dot{x}(t) \quad = \quad A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))), \tag{1.2}$$

provided that (1.1) admits an exponential dichotomy, the matrices $A(\cdot)$ and $A_0(\cdot)$ and $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ are such that

(A1) There exist positive constants M and M_0 such that

$$\sup_{-\infty<t<+\infty} ||A(t)|| \leq M \quad \text{and} \quad \sup_{-\infty<t<+\infty} ||A_0(t)|| \leq M_0,$$

where $||\cdot||$ denotes a matrix norm,

(A2) there exists a positive constant μ such that

$$|f(t,x,y)| \leq \mu \quad \text{for any} \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,$$

where $|\cdot|$ denotes a vector norm.

(A3) there exist positive constants ℓ_1 and ℓ_2 such that if $x, x', y, y' \in \mathbb{R}^n$

$$|f(t,x,y) - f(t,x',y')| \leq \ell_1|x-x'| + \ell_2|y-y'| \quad \text{for any} \quad t\in\mathbb{R}.$$

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The study of systems with piecewise constant arguments begin with Myshkis [20], which considers $\gamma(t) = [t]$ (the integer part), this case and other variations were usually known as DEPCA (Differential Equations with Piecewise Constant Argument) in the literature. A generalization was made by Akhmet [1], which introduces the DEPCAG (Differential Equations with Piecewise Constant Generalized Argument) by considering two sequences $\{t_i\}_{i\in\mathbb{Z}}$ and $\{\zeta_i\}_{i\in\mathbb{Z}}$, which satisfy:

(B1) $t_i < t_{i+1}$ and $t_i \leq \zeta_i \leq t_{i+1}$ for any $i \in \mathbb{Z}$,
(B2) $t_i \to \pm\infty$ as $i \to \pm\infty$,
(B3) $\gamma(t) = \zeta_i$ for $t \in [t_i, t_{i+1})$,
(B4) there exists a constant $\theta > 0$ such that

$$t_{i+1} - t_i = \theta_i \leq \theta, \quad \text{for any} \quad i \in \mathbb{Z}.$$

There exists an intensive theoretical research in DEPCAG equations (see, for instance, the monographies [1, 11, 34]), which has been accompanied with applications in engineering, life sciences and numerical analysis of ODE–DDE systems [3, 9, 13, 14, 21, 26, 29, 32, 33, 37].

1.1. **Topological equivalence.** The concept of topological equivalence was introduced by Palmer in [22] and can be seen as a generalization of the well known Grobman–Hartman's theorem to a nonautonomous framework.

Definition 1. *The systems* (1.1) *and* (1.2) *are topologically equivalent if there exists a function* $H\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ *with the properties*

(i) *For each fixed* $t \in \mathbb{R}$, $u \mapsto H(t,u)$ *is an homeomorphism of* $\mathbb{R}^n$,
(ii) $H(t,u) - u$ *is bounded in* $\mathbb{R}\times\mathbb{R}^n$,
(iii) *if* $x(t)$ *is a solution of* (1.2), *then* $H[t,x(t)]$ *is a solution of* (1.1),

In addition, the function $L(t,u) = H^{-1}(t,u)$ *has properties* (i)–(iii) *also.*

The concept of strongly topologically equivalence was introduced by Shi and Xiong [31], who realized that, in several examples of topological equivalence, the maps $u \mapsto H(t,u)$ and $u \mapsto L(t,u)$ could have properties sharper than continuity.

Definition 2. *The systems* (1.1) *and* (1.2) *are strongly topologically equivalent if they are topologically equivalent and* H *and* L *are uniformly continuous for all* t.

1.2. **Exponential dichotomy.** The exponential dichotomy property can be viewed as a generalization of the hiperbolicity property of linear autonomous systems and plays an important role in the study of linear systems.

Definition 3. *(see* [10]*) The system*

$$x' = A(t)x \tag{1.3}$$

has an $\widetilde{\alpha}$*–exponential dichotomy if there exists a projection* P *(*$P^2 = P$*) and two constants* $\tilde{K} \geq 1, \tilde{\alpha} > 0$ *such that* $\Phi(t)$, *the Cauchy matrix of* (1.3), *satisfies*

$$\begin{cases} ||\Phi(t)P\Phi^{-1}(s)|| \leq \tilde{K}e^{-\tilde{\alpha}(t-s)} & \text{if} \quad t \geq s \\ ||\Phi(t)(I-P)\Phi^{-1}(s)|| \leq \tilde{K}e^{-\tilde{\alpha}(s-t)} & \text{if} \quad s > t. \end{cases} \tag{1.4}$$

There are not a univoque definition of exponential dichotomy in a DEPCAG framework. The main dificulty is that the transition matrix $Z(t,\tau)$ of (1.1) can be constructed only when certain technical conditions are satisfied (see section 2). We will consider two definitions:

Definition 4. (*Akhmet* [2, 3]) *The linear DEPCAG* (1.1) *has an α–exponential dichotomy on* $(-\infty,\infty)$ *if there exists a projection P and some constants $K \geq 1$ and $\alpha > 0$, such that its transition matrix $Z(t,s)$ verifies*

$$||Z_p(t,s)|| \leq Ke^{-\alpha|t-s|} \tag{1.5}$$

where $Z_p(t,s)$ is defined by

$$Z_p(t,s) = \begin{cases} Z(t,0)PZ(0,s) & \text{if} \quad t \geq s \\ -Z(t,0)\{I-P\}Z(0,s) & \text{if} \quad s > t. \end{cases} \tag{1.6}$$

Definition 5. *The linear DEPCAG* (1.1) *has an exponential dichotomy on* $(-\infty,\infty)$ *if the system of difference equations*

$$y_{n+1} = Z(t_{n+1},t_n)y_n \tag{1.7}$$

has a discrete exponential dichotomy, which means that there exists a projection $\hat{P}$, $\hat{K} \geq 1$ and $0 < r < 1$ such that Y_n, the Cauchy matrix of (1.7) *verifies*

$$\begin{cases} ||Y_n\hat{P}Y_m^{-1}|| \leq \hat{K}r^{n-m} & \text{if} \quad n \geq m \\ ||Y_n\{I-\hat{P}\}Y_m^{-1}|| \leq \hat{K}r^{m-n} & \text{if} \quad m > n. \end{cases}$$

Remark 1. Notice that:

i) Definition 4 has been recently introduced by Akhmet in [2, 3] in order to study the existence of almost periodic solutions of almost periodic perturbations of (1.1). Definition 5 is employed in [7] with similar purposes. It is important to note that Definition 4 is oriented to a global treatement of (1.1) while Definition 5 allows the reduction to (1.7).

ii) A particular but distinguished case of Definition 5 restricted to $\gamma(t) = [t]$ was previously introduced by Papaschinopoulos [24, 25].

iii) Definitions 4 and 5 are independent and none implies the other. A deeper study about the relationship between definitions above remains to be done. Some preliminar comparative examples are presented in [7].

1.3. **Background and developments.** The seminal paper of Palmer [22] proves that if (1.3) has an exponential dichotomy (1.4) and the perturbed system

$$x' = A(t)x + f(t,x), \tag{1.8}$$

satisfies

$$|f(t,x)| \leq \tilde{\mu} \quad \text{and} \quad |f(t,x_1) - f(t,x_2)| \leq \tilde{\ell}|x_1 - x_2| \quad \text{for all} \quad t,x,x_1,x_2, \tag{1.9}$$

then (1.3) and (1.8) are topologically equivalent provided that $2\tilde{\ell}\tilde{K} \leq \tilde{\alpha}$.

Palmer's result of topological equivalence has been generalized in several directions: ordinary differential equations [5, 15, 16, 31], difference equations [4, 6, 18, 23], impulsive equations [17, 36] and time-scales systems [30, 35].

In a DEPCA framework, there exists a result of topological equivalence obtained by G. Papaschinopoulos [24, Proposition 1] for the special case $\gamma(t) = [t]$ by following the lines of the Palmer's work and introducing its *ad-hoc* definition of exponential dichotomy for (1.1).

This work generalizes the topological equivalence result of [24] in several directions. Firstly, we consider a general piecewise constant argument of advanced/delayed type. Secondly, we obtain conditions for strongly and Hölder strongly topological

equivalence. Thirdly, instead of Papaschinopoulos's definition of exponential dichotomy of (1.1), we use Definition 4, which allows a global treatment and considers limit cases that cannot be trated by the Papaschinopoulos's definition. More technical generalizations will be explained later.

1.4. **Outline.** Section 2 introduces technical notation, recalls the variation of parameters formula presented in [27] and states a result (Theorem 1) about existence and uniqueness of bounded solutions for bounded perturbations of (1.1). Section 3 states the two main results (Theorems 2 and 3) of stronlgy topological equivalence. Sections 4 and 5 state technical intermediate results. The proof of the main results is finished in section 6.

2. Technical preliminaries

In order to make the article self–contained, we will recall some previous notation and results obtained in [27].

Definition 6. [3, 34] *A continuous function $u(t)$ is solution of* (1.1) *or* (1.2) *if:*

(i) *The derivative $u'(t)$ exists at each point $t\in\mathbb{R}$ with the possible exception of the points t_i, $i\in\mathbb{Z}$, where the one side derivatives exists;*
(ii) *The equation is satisfied for $u(t)$ on each interval (t_i,t_{i+1}), and it holds for the right derivative of $u(t)$ at the points t_i.*

Without loss of generality, we will assume that the Cauchy matrix of (1.3) satisfies $\Phi(0)=I$. As usual, the transition matrix related to $A(t)$ will be denoted by $\Phi(t,s)=\Phi(t)\Phi^{-1}(s)$.

In [1, 27], the following $n\times n$ matrices are introduced:

$$J(t,\tau)=I+\int_{\tau}^{t}\Phi(\tau,s)A_0(s)\,ds, \tag{2.1}$$

$$E(t,\tau)=\Phi(t,\tau)+\int_{\tau}^{t}\Phi(t,s)A_0(s)\,ds=\Phi(t,\tau)J(t,\tau). \tag{2.2}$$

Given a set of $n\times n$ matrices $\mathcal{Q}_k$ $(k=1,\ldots,m)$, we will consider the product in the backward and forward sense as follows:

$$\prod_{k=1}^{\leftarrow m}\mathcal{Q}_k=\left\{\begin{array}{lll} \mathcal{Q}_m\cdots\mathcal{Q}_2\mathcal{Q}_1 & \text{if} & m\geq 1\\ I & \text{if} & m<1.\end{array}\right.$$

and

$$\prod_{k=1}^{\rightarrow m}\mathcal{Q}_k=\left\{\begin{array}{lll} \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m & \text{if} & m\geq 1\\ I & \text{if} & m<1.\end{array}\right.$$

2.1. **Notation and facts related to the sequences $\{t_i\}$ and $\{\zeta_i\}$.** The following notation will be useful:

- For any $k\in\mathbb{Z}$, we define $I_k=[t_k,t_{k+1})$, $I_k^+=[t_k,\zeta_k]$ and $I_k^-=[\zeta_k,t_{k+1})$.
- For any $t\in\mathbb{R}$, we define $i(t)\in\mathbb{Z}$ as the unique integer such that $t\in I_i=[t_i,t_{i+1})$.
- The number of the terms of the sequence $\{t_i\}$ contained in the interval (τ,t) will be denoted by $i(\tau,t)$.

- For any $k \in \mathbb{Z}$ and any matrix $t \mapsto Q(t) \in M_n(\mathbb{R})$, we define the numbers:

$$\rho_k^+(Q) = \exp\Big(\int_{t_k}^{\zeta_k} |Q(s)|\,ds\Big), \quad \text{and} \quad \rho_k^-(Q) = \exp\Big(\int_{\zeta_k}^{t_{k+1}} |Q(s)|\,ds\Big).$$

Some examples of functions $\gamma(t)$ and its corresponding sequences $\{t_k\}$ and $\{\zeta_k\}$ satisfying **(B1)**–**(B4)** are summarized in the following table (see [34] for details):

$\gamma(t)$	$\{t_k\}$	$\{\zeta_k\}$	Restrictions	Comments
$[t]$	k	k		completely delayed
$[t-j]$	k	$k-j$	$j \in \mathbb{Z}^+$	completely delayed
$[t+j]$	k	$k+j$	$j \in \mathbb{Z}^+$	completely advanced
$[t+1/2]$	k	$k+1/2$		advanced/delayed
$2[(t+1)/2]$	$2k$	$2k+1$		advanced/delayed
$\alpha h[t/(\alpha h)]$	$k\alpha h$	$k\alpha h$	$\alpha > 0, h > 0$	completely delayed
$m[(t+j)/m]$	$mk-j$	mk	$m > j > 0$	advanced/delayed

It is interesting to point out that the last two examples are functions $t \mapsto \gamma(t)$ employed in DEPCAG equations while the previous ones are classical examples used in DEPCA equations. The qualitative difference is that, in the first examples, the sequences $\{t_k\}$ and $\{\zeta_k\}$ are strictly determined, while in last cases they are dependent of the parameters α and m respectively, which induce α–parameter (resp. m–parameter) dependent families of sequences $\{t_k\}$ and $\{\zeta_k\}$.

Lemma 2.1. *For any s and t, it follows that*

$$|\gamma(s) - t| \leq \theta + |t - s|, \tag{2.3}$$

where θ is the same stated in **(B4)**.

Proof. As $s \in [t_{i(s)}, t_{i(s)+1})$, it follows that $\gamma(s) = \zeta_{i(s)}$. Now **(B1)** implies that

$$t_{i(s)} - t_{i(s)+1} \leq \zeta_{i(s)} - t_{i(s)+1} < \gamma(s) - s < \zeta_{i(s)} - t_{i(s)} < t_{i(s)+1} - t_{i(s)}$$

and **(B4)** implies that $|\gamma(s) - s| \leq \theta$.

Finally, (2.3) follows from $|\gamma(s) - t| \leq |\gamma(s) - s| + |s - t|$. □

2.2. **Complementary assumptions about A and A_0.** Throughout this article, we will asume that

(C) There exists $\nu^+ > 0$ and $\nu^- > 0$ such that the matrices $A(t)$ and $A_0(t)$ satisfy the properties:

$$\sup_{k\in\mathbb{Z}} \rho_k^+(A) \ln \rho_k^+(A_0) \leq \nu^+ < 1 \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \rho_k^-(A) \ln \rho_k^-(A_0) \leq \nu^- < 1. \tag{2.4}$$

Notice that **(A1)** and **(B4)** imply that

$$\rho(A) = \sup_{k\in\mathbb{Z}} \rho_k^+(A)\rho_k^-(A) < +\infty. \tag{2.5}$$

An important consequence of **(C)** is the following result:

Lemma 2.2. [27, Lemma 4.3] *If* (2.4) *is verified, it follows that*

$$|\Phi(t,s)| \leq \rho(A) \quad \text{for any} \quad t, s \in I_i.$$

and $J(t,s)$ is nonsingular for any $t, s \in I_i$.

2.3. **Variation of parameters formula.** Throughout the rest of this section, it will be assumed that **(A)**,**(B)** and **(C)** are satisfied.

A distinguished feature of DEPCAG systems is that their solutions could be noncontinuable in several cases. In this context, the condition **(C)** is introduced in [27] in order to provide sufficient conditions ensuring the continuability of the solutions of (1.1) to $(-\infty,+\infty)$. Furthermore, condition **(C)** and Lemma 2.2 imply that $J(t,s)$ and $E(t,s)$ are nonsingular for any $t,s\in I_i$, which allow to construct the transition matrix for (1.1) and to derive the variation of parameters formula.

Proposition 1. [27, p.239] *For any $t\in I_j,\tau\in I_i$, the solution of* (1.1) *with $z(\tau)=\xi$ is defined by*

$$z(t)=Z(t,\tau)\xi,$$

where $Z(t,\tau)$ is defined by

$$\begin{aligned} Z(t,\tau) &= E(t,\zeta_j)E(t_j,\zeta_j)^{-1}\prod_{k=i+2}^{\overleftarrow{j}} E(t_k,\gamma(t_{k-1}))E(t_{k-1},\gamma(t_{k-1}))^{-1} \\ &\quad E(t_{i+1},\gamma(\tau))E(\tau,\gamma(\tau))^{-1}, \end{aligned} \tag{2.6}$$

when $t>\tau$ and by

$$\begin{aligned} Z(t,\tau) &= E(t,\zeta_j)E(t_j,\zeta_j)^{-1}\prod_{k=i+2}^{\overrightarrow{j}} E(t_k,\gamma(t_k))E(t_k,\gamma(t_{k-1}))^{-1} \\ &\quad E(t_i,\gamma(\tau))E(\tau,\gamma(\tau))^{-1}, \end{aligned} \tag{2.7}$$

when $t<\tau$.

Remark 2. A direct consequence of Proposition 1 is that the operator $Z(\cdot,\cdot)$ verifies

$$Z(t,\tau)Z(\tau,s)=Z(t,s) \quad \text{and} \quad Z(t,s)=Z(s,t)^{-1}. \tag{2.8}$$

In addition, by using the facts

$$E(\tau,\tau)=I \quad \text{and} \quad \frac{\partial E}{\partial t}(t,\tau)=A(t)E(t,\tau)+A_0(t)$$

combined with Proposition 1, we can deduce that:

$$\frac{\partial Z}{\partial t}(t,\tau)=A(t)Z(t,\tau)+A_0(t)Z(\gamma(t),\tau). \tag{2.9}$$

Proposition 2 (Th. 3.1, [27])**.** *For any $j>i$, $t\in I_j$ and $\tau\in I_i$, the solution of*

$$\dot{x}(t) \;=\; A(t)x(t)+A_0(t)x(\gamma(t))+g(t), \tag{2.10}$$

with $z(\tau)=\xi$ is defined by

$$\begin{aligned} x(t) &= Z(t,\tau)\xi+\int_{\tau}^{\zeta_i} Z(t,\tau)\Phi(\tau,s)g(s)\,ds+\sum_{r=i+1}^{j}\int_{t_r}^{\zeta_r} Z(t,t_r)\Phi(t_r,s)g(s)\,ds \\ &\quad +\sum_{r=i}^{j-1}\int_{\zeta_r}^{t_{r+1}} Z(t,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds \\ &\quad +\operatorname{Sgn}(t-\zeta_j)\int_{\min\{\zeta_j,t\}}^{\max\{\zeta_j,t\}}\Phi(t,s)g(s)\,ds, \end{aligned}$$

when $\tau\in I_i^+=[t_i,\zeta_i)$.

It is important to emphasize that, when we consider any interval $I_k = [t_k, t_{k+1})$, we have the corresponding system of difference equations

$$\begin{aligned} x(t_{n+1}) &= Z(t_{n+1}, t_n)x(t_n) + \int_{t_n}^{\zeta_n} Z(t_{n+1}, t_n)\Phi(t_n, s)g(s)\, ds \\ &\quad + \int_{\zeta_n}^{t_{n+1}} \Phi(t_{n+1}, s)g(s)\, ds, \end{aligned}$$

which plays a key role to obtain the solution of (2.10). This non-homogeneous difference equation justifies Definition 5. The most studied case is $t_n = n$, that arises when $\gamma(t) = [t]$.

Lemma 2.3. *If the linear DEPCAG* (1.1) *has an α–exponential dichotomy on* $(-\infty, \infty)$*, then the unique solution bounded on* $(-\infty, +\infty)$ *is the null solution.*

Proof. By following the lines of Coppel [10], let us note that (1.5) is equivalent to:

$$|Z(t, 0)P\nu| \leq Ke^{-\alpha(t-s)}|Z(s, 0)P\nu| \quad \text{for} \quad t \geq s$$

$$|Z(t, 0)(I - P)\nu| \leq Ke^{-\alpha(s-t)}|Z(s, 0)(I - P)\nu| \quad \text{for} \quad t < s.$$

for any arbirtary $\nu \in \mathbb{R}^n$. Let us assume that P has rank k, then, the first inequality says that there is a k–dimensional vector space of initial conditions, such that it corresponding solutions converge to 0 when $t \to +\infty$ (and are divergent when $s \to -\infty$). The second inequality says that there is a complementary $(n - k)$–dimensional space, whose corresponding solutions are divergent when $s \to +\infty$ (and converge to 0 when $t \to -\infty$). The conclusion follows easily from those properties. □

Now, let us define the Green function corresponding to (1.1) in the interval $(-\infty, \infty)$:

Definition 7. *Given $t \in (\zeta_j, t_{j+1})$ and $Z_p(t, \tau)$ introduced in* (1.6)*, let us define*

$$\widetilde{G}(t, s) = \begin{cases} Z_p(t, t_r)\Phi(t_r, s) & \text{if} \quad s \in [t_r, \zeta_r) \quad \text{for any} \quad r \in \mathbb{Z}, \\ Z_p(t, t_{r+1})\Phi(t_{r+1}, s) & \text{if} \quad s \in [\zeta_r, t_{r+1}) \quad \text{for any} \quad r \in \mathbb{Z} \setminus \{j\}, \\ \Phi(t, s) & \text{if} \quad s \in [\zeta_j, t), \\ 0 & \text{if} \quad s \in [t, t_{j+1}), \end{cases}$$

and if $t \in [t_j, \zeta_j]$

$$\widetilde{G}(t, s) = \begin{cases} Z_p(t, t_r)\Phi(t_r, s) & \text{if} \quad s \in [t_r, \zeta_r) \quad \text{for any} \quad r \in \mathbb{Z} \setminus \{j\}, \\ Z_p(t, t_{r+1})\Phi(t_{r+1}, s) & \text{if} \quad s \in [\zeta_r, t_{r+1}) \quad \text{for any} \quad r \in \mathbb{Z}, \\ 0 & \text{if} \quad s \in [t_j, t), \\ -\Phi(t, s) & \text{if} \quad s \in [t, \zeta_j), \end{cases}$$

It is important to observe that $\widetilde{G}$ takes into account delayed and advanced intervals.

Proposition 3. *If the DEPCAG* (1.1) *has an α–exponential dichotomy* (1.5)*, then $\widetilde{G}$ satisfies*

$$|\widetilde{G}(t, s)| \leq K\rho^* e^{-\alpha|t-s|}, \quad \text{where} \quad \rho^* = \rho(A)e^{\alpha\theta}. \tag{2.11}$$

Proof. Without loss of generality, let us assume that $\zeta_j < t < t_{j+1}$. If $s \notin [t_j, t_{j+1}]$, there exists $r \neq j$ such that either $s \in [t_r, \zeta_r]$ or $s \in [\zeta_r, t_{j+1}]$.

Firstly, if $s \in [t_r, \zeta_r]$ and $j > r$, we have that $t > t_r$. This fact, combined with Lemma 2.2, eq.(1.5) and Definition 7 imply

$$\begin{array}{rcl} |\widetilde{G}(t,s)| & = & |Z_p(t,t_r)\Phi(t_r,s)| \\ & \leq & Ke^{-\alpha(t-t_r)}\rho(A) \\ & \leq & Ke^{-\alpha(t-s)}\rho(A)e^{\alpha\theta}. \end{array}$$

Secondly, if $s \in [t_r, \zeta_r]$ and $j < r$, we have that $t \leq t_r \leq s$. As before, we can deduce that

$$\begin{array}{rcl} |\widetilde{G}(t,s)| & \leq & Ke^{-\alpha(t_r-t)}\rho(A) \\ & \leq & Ke^{-\alpha(t_r-s)}\rho(A) \\ & \leq & Ke^{-\alpha(t-s)}\rho^*. \end{array}$$

The reader can obtain similar estimations in the case $s \in [\zeta_r, t_{j+1}]$. Finally, if $s \in I_j$, by using $K \geq 1$ combined with Lemma 2.2, we can deduce that

$$\begin{array}{rcl} |\widetilde{G}(t,s)| & \leq & \rho(A) \\ & \leq & K\rho(A)e^{\alpha|t-s|}e^{-\alpha|t-s|} \\ & \leq & Ke^{-\alpha|t-s|}\rho^*, \end{array}$$

and the Lemma follows. □

Remark 3. Notice that if θ is arbitrarily small, then ρ^* is arbitrarily close to one and equation (2.11) is close to

$$|\widetilde{G}(t,s)| \leq Ke^{-\alpha|t-s|},$$

which is the estimation of the Green's function in the ODE case.

Theorem 1. *If DEPCAG* (1.1) *has an α–exponential dichotomy and the series*

$$\sum_{r=-\infty}^{k} PZ(0,t_r)\int_{t_r}^{\zeta_r}\Phi(t_r,s)\,ds, \qquad \sum_{r=-\infty}^{k} PZ(0,t_{r+1})\int_{\zeta_r}^{t_{r+1}}\Phi(t_{r+1},s)\,ds, \tag{2.12}$$

and

$$\sum_{r=k}^{+\infty}(I-P)Z(0,t_r)\int_{t_r}^{\zeta_r}\Phi(t_r,s)\,ds,\ \sum_{r=k}^{+\infty}(I-P)Z(0,t_{r+1})\int_{\zeta_r}^{t_{r+1}}\Phi(t_{r+1},s)\,ds, \tag{2.13}$$

are absolutely convergent for any integer k, then for each bounded function $t \mapsto g(t)$, the system (2.10) *has a unique solution bounded on $\mathbb{R}$, defined by*

$$x_g^*(t) = \int_{-\infty}^{\infty}\widetilde{G}(t,s)g(s)\,ds$$

and the map $g \mapsto x_g$ is Lipschitz satisfying

$$|x_g^*|_\infty \leq \frac{2K\rho^*}{\alpha}|g|_\infty,$$

Proof. Without loss of generality, let us assume that $0 \in [t_i, \zeta_i)$ and $t \in [\zeta_j, t_{j+1})$ with $j > i$.

Step 1: We will prove that

$$\begin{aligned} x_g^*(t) &= \sum_{r=-\infty}^{j} \int_{t_r}^{\zeta_r} Z(t,0)PZ(0,t_r)\Phi(t_r,s)g(s)\,ds \\ &+ \sum_{r=-\infty}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t,0)PZ(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds \\ &- \sum_{r=j+1}^{+\infty} \int_{t_r}^{\zeta_r} Z(t,0)(I-P)Z(0,t_r)\Phi(t_r,s)g(s)\,ds \\ &- \sum_{r=j+1}^{+\infty} \int_{\zeta_r}^{t_{r+1}} Z(t,0)(I-P)Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds + \int_{\zeta_j}^{t} \Phi(t,s)g(s)\,ds, \end{aligned}$$

is a bounded solution of (2.10). Indeed, by using eq.(2.9) combined with $\int_{\zeta_j}^{\zeta_j} \Phi(\zeta_j,s)g(s)\,ds = 0$, it is easy to see that $t \mapsto x^*(t)$ is solution of (2.10). On the other hand, a careful reading of Definition 7 shows that

$$x_g^*(t) = \int_{-\infty}^{+\infty} \widetilde{G}(t,s)g(s)\,ds,$$

and the boundedness follows from Proposition 3.
Step 2: We will prove that $x_g^*(t)$ is the unique bounded solution of (2.10). Indeed, let $t \mapsto x(t)$ be a bounded solution. By using Proposition 2 with $\tau = 0$, we have that

$$\begin{aligned} x(t) &= Z(t,0)x(0) + \int_0^{\zeta_i} Z(t,0)\Phi(0,s)g(s)\,ds + \sum_{r=i+1}^{j} \int_{t_r}^{\zeta_r} Z(t,t_r)\Phi(t_r,s)g(s)\,ds \\ &+ \sum_{r=i}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds + \int_{\zeta_j}^{t} \Phi(t,s)g(s)\,ds, \end{aligned}$$

which can be written as

$$\begin{aligned} x(t) &= Z(t,0)\Big\{x(0) + \int_0^{\zeta_i} \Phi(0,s)g(s)\,ds + \sum_{r=i+1}^{j} P\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds \\ &+ \sum_{r=i}^{j-1} P\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds\Big\} + \int_{\zeta_j}^{t} \Phi(t,s)g(s)\,ds \\ &+ Z(t,0)\Big\{\sum_{r=i+1}^{j} (I-P)\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds \\ &+ \sum_{r=i}^{j-1} (I-P)\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds\Big\}. \end{aligned}$$

By using (2.12)–(2.13), we have that

$$\begin{aligned}
\sum_{r=i+1}^{j} P\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds &= \sum_{r=-\infty}^{j} P\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds - \\
&\quad \sum_{r=-\infty}^{i} P\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds, \\
\sum_{r=i}^{j-1} P\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds &= \sum_{r=-\infty}^{j-1} P\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds - \\
&\quad \sum_{r=-\infty}^{i-1} P\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds
\end{aligned}$$

and

$$\begin{aligned}
\sum_{r=i+1}^{j} (I-P)\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds &= \sum_{r=i+1}^{+\infty} (I-P)\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds - \\
&\quad \sum_{r=j+1}^{+\infty} (I-P)\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds.
\end{aligned}$$

Moreover, notice that

$$\sum_{r=i}^{j-1} (I-P)\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds =$$

$$\sum_{r=i,r\neq j}^{+\infty} (I-P)\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds - \sum_{r=j+1}^{+\infty} (I-P)\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds,$$

and we can see that the bounded solution $x(t)$ can be written as follows

$$x(t) = Z(t,0)\{x(0)+x_1+x_2\} + x_g^*(t),$$

where

$$\begin{aligned}
x_1 &= \int_0^{\zeta_i} \Phi(0,s)g(s) - \sum_{r=-\infty}^{i} P\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds \\
&\quad - \sum_{r=-\infty}^{i-1} P\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds
\end{aligned}$$

and

$$\begin{aligned}
x_2 &= \sum_{r=i+1}^{+\infty} (I-P)\int_{t_r}^{\zeta_r} Z(0,t_r)\Phi(t_r,s)g(s)\,ds \\
&\quad + \sum_{r=i,r\neq j}^{+\infty} (I-P)\int_{\zeta_r}^{t_{r+1}} Z(0,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds.
\end{aligned}$$

As $t \mapsto x_g^*(t)$ is a bounded solution of (2.10), we have that,

$$x(t) - x_g^*(t) = Z(t,0)\{x(0)+x_1+x_2\}$$

is a bounded solution of (1.1). Finally, Lemma 2.3 implies that $x(t) = x_g^*(t)$ and the uniqueness follows. □

Remark 4. Theorem 1 generalizes a classical result in the ODE case (see *e.g.*, [10, 19]) and has been previously proved by Akhmet and Yilmaz in [2, 3]. We point out that our proof was stated in [28] and has some technical differences: we follow a constructive approach to deduce the bounded solution, we consider the intervals $[t_r, \zeta_r)$ and $[\zeta_r, t_{r+1})$ instead of (ζ_r, ζ_{r+1}) and we work with different upper bounds of the transition matrix $Z(t_{r+1}, t_r)$.

Remark 5. The convergence of series (2.12)–(2.13) can be ensured by imposing additional properties to the sequence $\{t_r\}_r$. For example, by α–exponential dichotomy (1.5) combined with $Z(0,0) = I$ and Lemma 2.2, we conclude that

$$\sum_{r=k}^{+\infty} \left|(I-P)Z(0,t_{r+1}) \int_{\zeta_r}^{t_{r+1}} \Phi(t_{r+1},s)\right| ds \quad \leq \quad K\rho(A) \sum_{r=k}^{+\infty} e^{-\alpha|t_{r+1}|},$$

and the second series of (2.13) converges if the series $S_n = \sum\limits_{k=r}^{n} e^{-\alpha|t_{r+1}|}$ $(n > k)$ is convergent. Now, the convergence of S_n can be ensured in several cases. For example, if there exists $\bar{\theta} > 0$ such that

$$\bar{\theta} \leq t_{r+1} - t_r \quad \text{for any} \quad r \in \mathbb{Z},$$

we have that the series S_n is dominated by a geometric one. On the other hand, it is straightforward to see that, if there exists $C > 0$ such that

$$|t_{r+2}| - |t_{r+1}| \geq C \quad \text{or} \quad |t_r| \geq C|r| \quad \text{for any} \quad r > R$$

for any R arbitrarly large, then

$$\limsup_{r\to+\infty} e^{-\alpha(|t_{r+2}|-|t_{r+1}|)} < 1 \quad \text{or} \quad \limsup_{r\to+\infty} e^{-\frac{|t_{r+1}|}{r}} < 1,$$

which implies the convergence of the series by the ratio test or the radical test respectively.

Throughout this paper, we will assume that (2.12)–(2.13) are convergent.

3. Main Results

Theorem 2. *If* (1.1) *has a transition matrix* $Z(t,0)$ *satisfying the exponential dichotomy* (1.5)*, conditions* **(A)***,* **(B)** *and* **(C)** *are satisfied and*

$$2(\ell_1 + \ell_2)K\rho^* < \alpha, \tag{3.1}$$

$$F_1(\theta)(M_0 + \ell_2)\theta = v < 1, \quad \text{with} \quad F_1(\theta) = \frac{e^{(M+\ell_1)\theta} - 1}{(M+\ell_1)\theta}, \tag{3.2}$$

$$F_0(\theta)M_0\theta = \tilde{v} < 1, \quad \text{with} \quad F_0(\theta) = \frac{e^{M\theta} - 1}{M\theta}, \tag{3.3}$$

then (1.1) *and* (1.2) *are strongly topologically equivalent.*

Theorem 3. *If* (1.1) *has a transition matrix* $Z(t,0)$ *satisfying the exponential dichotomy* (1.5)*, conditions* **(A)***,* **(B)***,* **(C)***,* (3.1)–(3.3) *are satisfied and*

$$\alpha < M + \min\left\{\ell_1 + \frac{M_0 + \ell_2}{1 - v} e^{(M+\ell_1)\theta}, \frac{M_0}{1-\tilde{v}} e^{M\theta}\right\}, \tag{3.4}$$

then the systems (1.1) *and* (1.2) *are Hölder strongly topologically equivalent, namely, there exists constants* $C_1 > 1$,$D_1 > 1$,$C_2 \in (0,1)$ *and* $D_2 \in (0,1)$ *such that the maps* H *and* L *are Hölder continuous in the sense:*

$$|H(t,\xi) - H(t,\xi')| \leq C_1|\xi - \xi'|^{C_2} \quad \text{and} \quad |L(t,\nu) - H(t,\nu')| \leq D_1|\nu - \nu'|^{D_2}$$

for any couple (ξ,ξ') *and* (ν,ν') *verifying* $|\xi - \xi'| < 1$ *and* $|\nu - \nu'| < 1$.

Remark 6. As we stated in the introduction, we generalize Papaschinopoulos's result [24, Proposition 1] in several ways:

i) Theorems 2 and 3 consider a generic piecewise constant argument including the particular delayed case $\gamma(t) = [t]$,
ii) We obtain results sharper than topological equivalence, namely, strongly and Hölder topological equivalence,
iii) We use a recently introduced definition of exponential dichotomy,
iv) Our results don't need to assume that (1.3) has the exponential dichotomy (1.4) and allow limit cases as $A(t) = 0$ for any $t \in \mathbb{R}$,
v) The smallness of $A_0(\cdot)$ is not always necessary as in [24], for example a threshold between θ and M_0 ensuring $v < 1$ can be constructed.

Remark 7. Some comments about the conditions:

i) Inequality (3.1) is reminiscent of the contractivity condition stated by Palmer in [22]. Notice that if $\theta = 0$ (*i.e.*, $\rho^* = 1$) and $\ell_2 = 0$, then (3.1) becomes the Palmer's condition $2\ell_1 K < \alpha$.
ii) Inequalities (3.2) and (3.3) can be verified in several cases. For example, when θ is arbitrarily small. Indeed, notice that if $\theta \to 0^+$, then $F_0(\theta), F_1(\theta) \to 1$ and $v \approx (M_0 + \ell_2)\theta < 1$ (resp. $\tilde{v} \approx M_0\theta < 1$).
iii) In the section 2 of [8], it is proved that the inequality (3.2) implies the existence and uniqueness of the solutions of (1.2). Indeed, it will be useful to denote by $x(t,\tau,\xi)$ as the unique solution of (1.2) passing through ξ at $t = \tau$. By uniqueness of solutions of (1.2), we know that

$$x\big(s,t,x(t,\tau,\xi)\big) = x\big(s,\tau,\xi\big). \tag{3.5}$$

iv) Inequality (3.4) is related with the Hölder continuity in the classical strongly topological equivalence literature (see *e.g.*,[31]). In addition, it is always satisfied when $\alpha < M$.

The first byproduct states that strongly topological equivalence is an equivalence relation since the composition of homeomorphisms is an homeomorphism and its proof is left to the reader:

Corollary 1. *Let us consider the system*

$$\dot{x}(t) \quad = \quad A(t)x(t) + A_0(t)x(\gamma(t)) + h(t,x(t),x(\gamma(t))), \tag{3.6}$$

where A,A_0 *and* h *satisfy* **(A)** *and* $\gamma(\cdot)$ *satisfies* **(B)**. *If the assumptions of Theorem* 2 *(resp. Theorem* 3*) are satisfied, then* (1.2) *and* (3.6) *are strongly topologically equivalent (resp. Hölder strongly topologically equivalent).*

In the limit case $A_0(t) = 0$, we have that assumption **(C)** is always verified since $\ln\rho_k^+(A_0) = \ln\rho_k^-(A_0) = 0$. In addition, the linear DEPCAG system (1.1) becomes

the ODE system (1.3). Finally, we can see that $J(t,\tau)=I$, $E(t,\tau)=Z(t,\tau)=\Phi(t,\tau)$ and the Green function $\widetilde{G}(t,s)$ becomes:

$$G(t,s)=\left\{\begin{array}{rl}\Phi(t)P\Phi^{-1}(s) & \text{if} \quad t\geq s\\ -\Phi(t)(I-P)\Phi^{-1}(s) & \text{if} \quad s>t.\end{array}\right.$$

Now, it is easy to prove the following result:

Corollary 2. *If the system* (1.3) *has a Cauchy matrix* $\Phi(t)$ *satisfying the* $\tilde{\alpha}$*–exponential dichotomy* (1.4)*,* $A_0(t)=0$ *for any* t*, conditions* **(A)** *and* **(B)** *are satisfied in this context and*

$$2(\ell_1+\ell_2)\tilde{K}<\tilde{\alpha}, \tag{3.7}$$

$$F_1(\theta)\ell_2\theta=v_0<1, \tag{3.8}$$

then the systems (1.2) *and* (1.3) *are strongly topologically equivalent. In addition, if* $M>\tilde{\alpha}$*, then the systems* (1.2) *and* (1.3) *are Hölder strongly topologically equivalent.*

Finally, if $A(t)=0$, we have that (1.1)–(1.2) becomes

$$\dot{y}(t) \;=\; A_0(t)y(\gamma(t)), \tag{3.9}$$

$$\dot{x}(t) \;=\; A_0(t)x(\gamma(t))+f(t,x(t),x(\gamma(t))). \tag{3.10}$$

In this context, the reader can verify that $\Phi(t,\tau)=I$ and

$$J(t,\tau)=E(t,\tau)=I+\int_{\tau}^{t}A_0(s)\,ds.$$

In addition, $A(t)=0$ modify the corresponding definitions of $Z(t,s)$ and $\widetilde{G}(t,s)$ with $\rho^*=e^{\alpha\theta}$ and it is easy to prove:

Corollary 3. *If* (3.9) *has a transition matrix* $Z(t,0)$ *satisfying the exponential dichotomy* (1.5)*, conditions* **(A)** *and* **(B)** *are satisfied and*

$$2(\ell_1+\ell_2)Ke^{\alpha\theta}<\alpha, \tag{3.11}$$

$$\tilde{F}_1(\theta)(M_0+\ell_2)\theta=\tilde{v}_0<1, \quad \text{with} \quad \tilde{F}_1(\theta)=\frac{e^{\ell_1\theta}-1}{\ell_1\theta} \tag{3.12}$$

$$(M_0+\ell_2)\theta=\tilde{u}_0<1, \tag{3.13}$$

then (3.9) *and* (3.10) *are strongly topologically equivalent. In addition, if*

$$\alpha<\min\Big\{\ell_1+\frac{M_0+\ell_2}{1-\tilde{v}_0}e^{\ell_1\theta},\frac{M_0}{1-\tilde{u}_0}\Big\},$$

then the systems (1.2) *and* (1.3) *are Hölder strongly topologically equivalent.*

Proof. We only need to prove that **(C)** is satisfied with $A(t)=0$. Indeed, notice that $\rho(A)=1$ combined with **(A1)** and (3.13) imply that

$$\ln\rho_k^{\pm}(A_0)\leq M_0\theta<\tilde{u}_0<1$$

and (2.4) follows.

□

Remark 8. It is interesting to see that if $\theta \to 0^+$, then the (step) function $\gamma(t)$ converges uniformly to the identity function. This case is important in numerical approximation for solutions of differential delay equations (see *e.g.*, [13] for details). Moreover, the authors are working in the problem of the approximation of the solutions of the ODE systems

$$y' = A_0(t)y \tag{3.14}$$

and

$$x' = A_0(t)x + f(t,x,x), \tag{3.15}$$

uniformly on $(-\infty,+\infty)$ by solutions of (3.9)–(3.10) when $\theta \to 0^+$ and some preliminar results are presented in [12]. In this framework, these expected approximation results combined with corollaries 2 and 3 could help to deduce and generalize (by an alternative approach) the classical Palmer's result [22] about topological equivalence between (3.14) and (3.15). Notice that conditions **(A)**,**(B)**,**(C)** and inequalities (3.11)–(3.13) "converge" to those stated in Palmer's article. See Remarks 3 and 7.

4. Some Lemmas

Throughout this section, we will assume that the system (1.1) has a transition matrix $Z(t,0)$ satisfying the exponential dichotomy (1.5).

Lemma 4.1. *For any solution $x(t,\tau,\xi)$ of (1.2) passing through ξ at $t=\tau$, there exists a unique bounded solution $t \mapsto \chi(t;(\tau,\xi))$ of*

$$\dot{z}(t) \;=\; A(t)z(t) + A_0(t)z(\gamma(t)) - f(t,x(t,\tau,\xi),x(\gamma(t),\tau,\xi)). \tag{4.1}$$

Proof. By using Theorem 1 with $g(t) = -f(t,x(t,\tau,\xi),x(\gamma(t),\tau,\xi))$, we have that

$$\chi(t;(\tau,\xi)) = -\int_{-\infty}^{\infty} \widetilde{G}(t,s)f(s,x(s,\tau,\xi),x(\gamma(s),\tau,\xi))\,ds$$

is the unique bounded solution of (4.1). In addition, **(A2)** implies that $|\chi(t;(\tau,\xi))| \leq 2K\rho^*\mu\alpha^{-1}$. □

Remark 9. By uniqueness of solutions of (1.2) and equation (3.5) with $s=t$ and $s=\gamma(t)$, we know that

$$x\big(t,t,x(t,\tau,\xi)\big) = x\big(t,\tau,\xi\big) \quad \text{and} \quad x\big(\gamma(t),t,x(t,\tau,\xi)\big) = x\big(\gamma(t),\tau,\xi\big),$$

this fact implies that system (4.1) can be written as

$$\begin{aligned}\dot{z}(t) \;&=\; A(t)z(t) + A_0(t)z(\gamma(t)) \\ &\quad -f(t,x(t,t,x(t,\tau,\xi)),x(\gamma(t),t,x(t,\tau,\xi)))\end{aligned}$$

and Lemma 4.1 implies that

$$\chi(t;(\tau,\xi)) = \chi(t;(t,x(t,\tau,\xi))). \tag{4.2}$$

Lemma 4.2. *For any solution $y(t,\tau,\nu)$ of (1.1) passing through ν at $t=\tau$, there exists a unique bounded solution $t \mapsto \vartheta(t;(\tau,\nu))$ of*

$$\begin{aligned}\dot{w}(t) \;&=\; A(t)w(t) + A_0(t)w(\gamma(t)) \\ &\quad +f(t,y(t,\tau,\nu)+w(t),y(\gamma(t),\tau,\nu)+w(\gamma(t))).\end{aligned} \tag{4.3}$$

Proof. Let BC be the Banach space of bounded and continuous functions $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ with supremun norm. By Theorem 1, we know that the map $\Gamma\colon BC \to BC$:

$$\Gamma\varphi(t) = \int_{-\infty}^{\infty} \widetilde{G}(t,s) f(s, y(s,\tau,\nu) + \varphi(s), y(\gamma(s),\tau,\nu) + \varphi(\gamma(s)))\, ds,$$

is well defined. Now, notice that **(A3)** implies

$$\begin{aligned} |\Gamma\varphi(t) - \Gamma\phi(t)| &\leq \int_{\mathbb{R}} |\widetilde{G}(t,s)| \{\ell_1 |\varphi(s) - \phi(s)| + \ell_2 |\varphi(\gamma(s)) - \phi(\gamma(s))|\}\, ds \\ &\leq \frac{2K\rho^*}{\alpha}(\ell_1 + \ell_2)||\varphi - \phi||, \end{aligned}$$

and (3.1) implies that Γ is a contraction, having a unique fixed point satisfying

$$\begin{aligned} &\vartheta(t;(\tau,\nu)) = \\ &\int_{-\infty}^{+\infty} \widetilde{G}(t,s) f(s, y(s,\tau,\nu) + \vartheta(s;(\tau,\nu)), y(\gamma(s),\tau,\nu) + \vartheta(\gamma(s);(\tau,\nu)))\, ds \end{aligned} \tag{4.4}$$

and the reader can easily verify that is a bounded solution of (4.1). □

Remark 10. Similarly as in Remark 9, the reader can verify that

$$\vartheta(t;(\tau,\nu)) = \vartheta(t;(t, y(t,\tau,\nu))). \tag{4.5}$$

Lemma 4.3. *There exists a unique function $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, satisfying:*

(i) *$H(t,x) - x$ is bounded in $\mathbb{R} \times \mathbb{R}^n$,*
(ii) *For any solution $t \mapsto x(t)$ of* (1.2)*, then $t \mapsto H[t,x(t)]$ is a solution of* (1.1) *satisfying*

$$|H[t,x(t)] - x(t)| \leq 2\mu K\rho^*\alpha^{-1} \tag{4.6}$$

Proof. The proof will be decomposed in several steps.
Step 1) Existence of H: Let us define the function $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\begin{aligned} H(t,\xi) &= \xi + \chi(t;(t,\xi)) \\ &= \xi - \int_{-\infty}^{\infty} \widetilde{G}(t,s) f(s, x(s,t,\xi), x(\gamma(s),t,\xi))\, ds \end{aligned} \tag{4.7}$$

and **(A2)** implies $|H(t,\xi) - \xi| \leq 2\mu K\rho^*\alpha^{-1}$.

By replacing (t,ξ) by $(t, x(t,\tau,\xi))$ in (4.7), we have that

$$H[t, x(t,\tau,\xi)] = x(t,\tau,\xi) + \chi(t;(t, x(t,\tau,\xi)))$$

Now, by (4.2), we have

$$H[t, x(t,\tau,\xi)] = x(t,\tau,\xi) + \chi(t;(\tau,\xi)) \tag{4.8}$$

or equivalently

$$H[t, x(t,\tau,\xi)] = x(t,\tau,\xi) - \int_{-\infty}^{\infty} \widetilde{G}(t,s) f(s, x(s,\tau,\xi), x(\gamma(s),\tau,\xi))\, ds. \tag{4.9}$$

Finally, it is easy to verify that $t \mapsto H[t, x(t,\tau,\xi)]$ is solution of (1.1).
Step 2) Uniqueness of H: Let us suppose that there exists another map $\widetilde{H}$ satisfying properties (i) and (ii), this implies that $\widetilde{H}[t, x(t,\tau,\xi)]$ is solution of (1.1) and

$$\hat{z}(t,\xi) = \widetilde{H}[t, x(t,\tau,\xi)] - x(t,\tau,\xi)$$

is a bounded solution of (4.1). Nevertheless, as (4.1) has a unique bounded solution, we can conclude that $\hat{z}(t) = \chi(t;(\tau,\xi))$ and (4.8) implies that

$$\widetilde{H}[t,x(t,\tau,\xi)] = x(t,\tau,\xi) + \chi(t;(\tau,\xi)) = H[t,x(t,\tau,\xi)].$$

□

Lemma 4.4. *There exists a unique function $L\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, satisfying:*

(i) *$L(t,y) - y$ is bounded in $\mathbb{R}\times\mathbb{R}^n$,*

(ii) *For any solution $t\mapsto y(t)$ of (1.1), we have that $t\mapsto L[t,y(t)]$ is a solution of (1.2) verifying*

$$|L[t,y(t)] - y(t)| \leq 2\mu K\rho^*\alpha^{-1}. \tag{4.10}$$

Proof. The existence and uniqueness of the function L satisfying (i)–(ii) can be proved in a similar way. Indeed, L is defined by

$$L(t,\nu) = \nu + \vartheta(t;(t,\nu)),$$

where

$$\vartheta(t;(t,\nu)) = \int_{-\infty}^{\infty}\widetilde{G}(t,s)f(s,y(s,t,\nu)+\vartheta(s;(t,\nu)),y(\gamma(s),t,\nu)+\vartheta(\gamma(s);(t,\nu)))\,ds$$

As before, by using (4.5), for $y(t) = y(t,\tau,\nu)$ we can define

$$L[t,y(t)] = y(t,\tau,\nu) + \vartheta(t;(t,y(t,\tau,\nu))) = y(t,\tau,\nu) + \vartheta(t;(\tau,\nu)), \tag{4.11}$$

It will be useful to describe $L[t,y(t)]$ as follows

$$L[t,y(t)] \;=\; y(t) + \int_{-\infty}^{\infty}\widetilde{G}(t,s)f(s,L[s,y(s)],L[\gamma(s),y(\gamma(s))])\,ds. \tag{4.12}$$

□

Lemma 4.5. *For any solution $x(t)$ of (1.2) and $y(t)$ of (1.1) with fixed t, it follows that*

$$L[t,H[t,x(t)]] = x(t) \quad \text{and} \quad H[t,L[t,y(t)]] = y(t).$$

Proof. We will prove only the first identity. The other one can be deduced similarly and is given for the reader.

Let $t\mapsto x(t) = x(t,\tau,\xi)$ be a solution of (1.2). By using Lemma 4.3, we know that $H[t,x(t)]$ is solution of (1.1). Moreover, by Lemma 4.4, we can see that $t\mapsto J[t,x(t)] = L[t,H[t,x(t)]]$ is solution of (1.2). Notice that

$$J[t,x(t)] \;=\; H[t,x(t)] + \vartheta(t;(t,H[t,x(t)]))$$

where $t\mapsto \vartheta(t;(t,H[t,x(t)]))$ is the unique bounded solution of the system

$$\begin{aligned}\dot{w}(t) \;&=\; A(t)w(t) + A_0(t)w(\gamma(t))\\ &\quad +f(t,H[t,x(t)]+w(t),H[\gamma(t),x(\gamma(t))]+w(\gamma(t))).\end{aligned}$$

By using Lemma 4.4 with $H[t,x(t)]$ instead of $y(t)$, we have that

$$J[t,x(t)] \;=\; H[t,x(t)] + \int_{-\infty}^{\infty}\widetilde{G}(t,s)f(s,J[s,x(s)],J[\gamma(s),x(\gamma(s))])\,ds.$$

Upon inserting (4.9) in the identity above, we have that

$$J[t,x(t)]-x(t)=\int_{\mathbb{R}}\widetilde{G}(t,s)\{f(s,J[s,x(s)],J[\gamma(s),x(\gamma(s))])-f(s,x(s),x(\gamma(s)))\}\,ds,$$

which implies the inequality

$$|J[t,x(t)]-x(t)|\leq\frac{2K\rho^{*}}{\alpha}\big(\ell_1+\ell_2\big)|J[\cdot,x(\cdot)]-x(\cdot)|_{\infty}$$

and (3.1) implies that

$$J[t,x(t)]=L[t,H[t,x(t)]]=x(t).$$

□

The reader can notice (see also Definition 1) that the notation $H[\cdot,\cdot]$ and $L[\cdot,\cdot]$ is reserved to the case when H and L are respectively defined on solution of (1.2) and (1.1).

Lemma 4.6. *For any fixed t and any couple $(\xi,\nu)\in\mathbb{R}^n\times\mathbb{R}^n$, it follows that*

$$L(t,H(t,\xi))=\xi \tag{4.13}$$

and

$$H(t,L(t,\nu))=\nu. \tag{4.14}$$

Proof. By using Lemma 4.5, we have that

$$L[t,H[x(t,\tau,\xi)]=x(t,\tau,\xi)\quad\text{for any}\quad t\in\mathbb{R}.$$

Now, if we consider the particular case $\tau=t$, we obtain (4.13). The identity (4.14) can be deduced similarly. □

Remark 11. Notice that the maps $\xi\mapsto H(t,\xi)$ and $\nu\mapsto L(t,\nu)$ satisfy properties (ii) and (iii) of Definition 1, which is a consequence of Lemmatas 4.3–4.5. In addition, Lemma 4.6 says that $u\mapsto L(t,u)=H^{-1}(t,u)$ for any $t\in\mathbb{R}$. In consequence, the last step is to prove the uniform continuity of the maps, which will be made in the next two sections.

5. Continuity with respect to initial conditions

The following result generalizes the classical Gronwall's inequality to the DEPCAG framework:

Proposition 4. (*Gronwall's type inequality,* [8, Lemma 2.1]) *Let $u,\tilde{\eta}_i\colon\mathbb{R}\to[0,+\infty)$ $i=1,2$ be continuous functions and $\tilde{C}>0$. Suppose that for all $t\geq\tau$, the inequality*

$$u(t)\leq\tilde{C}+\int_{\tau}^{t}\{\tilde{\eta}_1(s)u(s)+\tilde{\eta}_2(s)u(\gamma(s))\}\,ds$$

holds. If

$$w=\sup_{i\in\mathbb{N}}\int_{t_i}^{\zeta_i}\tilde{\eta}_2(s)e^{\int_s^{\zeta_i}\tilde{\eta}_1(r)\,dr}\,ds<1,$$

then for any $t\geq\tau$ it follows that

$$u(t)\leq\tilde{C}\exp\Big(\int_{\tau}^{t}\tilde{\eta}_1(s)\,ds+\frac{1}{1-w}\int_{\tau}^{t}\Big[\tilde{\eta}_2(s)e^{\int_{t_i(s)}^{\gamma(s)}\tilde{\eta}_1(r)\,dr}\Big]\,ds\Big).$$

Similarly as in an ODE context, the Gronwall's inequality is a key tool in the proof of continuity with respect to the initial conditions:

Lemma 5.1. *Let $t \mapsto x(t,\tau,\xi)$ and $t \mapsto x(t,\tau,\xi')$ be the solutions of* (1.2) *passing respectively through ξ and ξ' at $t=\tau$. If* (3.2) *is verified, then it follows that*

$$|x(t,\tau,\xi') - x(t,\tau,\xi)| \leq |\xi - \xi'| e^{p_1|t-\tau|} \tag{5.1}$$

where p_1 is defined by

$$p_1 = \eta_1 + \frac{\eta_2 e^{\eta_1 \theta}}{1-v} \quad \text{with} \quad \eta_1 = M + \ell_1, \quad \eta_2 = M_0 + \ell_2 \tag{5.2}$$

and $v \in [0,1)$ is defined by (3.2).

Proof. Without loss of generality, we will assume that $t > \tau$, the case corresponding to $t < \tau$ can be proved similary and is left to the reader.

Firstly, let us consider the case $t_i < \tau < t < t_{i+1}$ for some $i \in \mathbb{Z}$, then notice that **(A1)** and **(A3)** imply

$$\begin{aligned} |x(t,\tau,\xi') - x(t,\tau,\xi)| \leq & \; |\xi - \xi'| + \int_{\tau}^{t} \big\{ \eta_1 |x(s,\tau,\xi') - x(s,\tau,\xi)| \\ & + \eta_2 |x(\gamma(s),\tau,\xi') - x(\gamma(s),\tau,\xi)| \big\} \, ds. \end{aligned}$$

As (3.2) implies that

$$\int_{t_i}^{\zeta_i} \eta_2 e^{\eta_1(\zeta_i - s)} \, ds = \frac{\eta_2}{\eta_1} \Big(e^{\eta_1(\zeta_i - t_i)} - 1 \Big) \leq v < 1,$$

then Proposition 4 combined with $\zeta_i - t_i \leq \theta$ for any $i \in \mathbb{Z}$ imply (5.1) for any $t \in (\tau, t_{i+1}]$. In particular, at $t = t_{i+1}$, we have that

$$|x(t_{i+1},\tau,\xi') - x(t_{i+1},\tau,\xi)| \leq |\xi' - \xi| \exp\left(\left\{ \eta_1 + \frac{\eta_2 e^{\eta_1 \theta}}{1-v} \right\} (t_{i+1} - \tau) \right). \tag{5.3}$$

Secondly, let us consider $t \in (t_{i+1}, t_{i+2}]$ and notice that uniqueness of the solutions imply

$$x(t, t_{i+1}, x(t_{i+1},\tau,\xi)) = x(t,\tau,\xi), \tag{5.4}$$

and

$$x(\gamma(t), t_{i+1}, x(t_{i+1},\tau,\xi)) = x(\gamma(t),\tau,\xi), \tag{5.5}$$

As in the previous step, we can observe that

$$\begin{aligned} |x(t,\tau,\xi') - x(t,\tau,\xi)| \leq & \; |x(t_{i+1},\tau,\xi') - x(t_{i+1},\tau,\xi)| \\ & + \int_{t_{i+1}}^{t} \big\{ \eta_1 |x(s,\tau,\xi') - x(s,\tau,\xi)| \\ & + \eta_2 |x(\gamma(s),\tau,\xi') - x(\gamma(s),\tau,\xi)| \big\} \, ds \end{aligned} \tag{5.6}$$

for any $t \in (t_{i+1}, t_{i+2}]$. By applying the Gronwall's type inequality to (5.6) combined with (5.3) and (5.4), we can deduce that

$$\begin{aligned} |x(t,\tau,\xi') - x(t,\tau,\xi)| & \leq |x(t_{i+1},\tau,\xi') - x(t_{i+1},\tau,\xi)| \exp\left(\left\{ \eta_1 + \tfrac{\eta_2 e^{\eta_1 \theta}}{1-v} \right\} (t - t_{i+1}) \right) \\ & \leq |\xi' - \xi| \exp\left(\left\{ \eta_1 + \tfrac{\eta_2 e^{\eta_1 \theta}}{1-v} \right\} (t - \tau) \right) \end{aligned}$$

for any $t \in (t_{i+1}, t_{i+2}]$ and the reader can verify that (5.1) is valid for any $t \geq \tau$ in a recursive way. □

The next results are similar to the previous one and its proof is left to the reader.

Lemma 5.2. *Let $t \mapsto y(t,\tau,\nu)$ and $t \mapsto y(t,\tau,\nu')$ be the solutions of* (1.1) *passing respectively through ν and ν' at $t=\tau$. If* (3.3) *is satisfied, then:*

$$|y(t,\tau,\nu') - y(t,\tau,\nu)| \leq |\nu - \nu'| e^{p_2|t-\tau|} \quad \text{with} \quad p_2 = M + \frac{M_0 e^{M\theta}}{1-\tilde{v}}, \tag{5.7}$$

where $\tilde{v} \in [0,1)$ is defined by (3.3).

Lemma 5.3. *Let $t \mapsto x(t,\tau,\xi)$ and $t \mapsto x(t,\tau,\xi')$ (resp. $t \mapsto y(t,\tau,\nu)$ and $t \mapsto y(t,\tau,\nu')$) be the solutions of* (3.10) *(resp.*(3.9)*) passing through ξ and ξ' (resp. ν and ν') at $t=\tau$. If* (3.12) *and* (3.13) *are satisfied, then:*

$$|x(t,\tau,\xi') - x(t,\tau,\xi)| \leq |\xi - \xi'| e^{\tilde{p}_1|t-\tau|} \quad \text{with} \quad \tilde{p}_1 = \ell_1 + \frac{(M_0+\ell_2)e^{\ell_1\theta}}{1-\tilde{v}_0}, \tag{5.8}$$

and

$$|y(t,\tau,\nu') - y(t,\tau,\nu)| \leq |\nu - \nu'| e^{\tilde{p}_2|t-\tau|} \quad \text{with} \quad \tilde{p}_2 = \frac{M_0}{1-\tilde{u}_0}, \tag{5.9}$$

where $\tilde{v}_0 \in [0,1)$ and $\tilde{u}_0 \in [0,1)$ are respectively defined by (3.12) *and* (3.13).

6. Proof of main results

6.1. **Proof of Theorem 2.** As stated in Remark 11, we only have to prove that the maps $\xi \mapsto H(t,\xi)$ and $\nu \mapsto L(t,\nu)$ defined in the section 4 are uniformly continuous.

Lemma 6.1. *The map $\xi \to H(t,\xi) = \xi + \chi(t;(t,\xi))$ is uniformly continuous for any t.*

Proof. As the identity is uniformly continuous, we only need to prove that the map $\xi \to \chi(t;(t,\xi))$ is uniformly continuous.

Let ξ and ξ' be two initial conditions of (1.2). Notice that (4.7) allows to say that

$$\begin{aligned}
\chi(t;(t,\xi)) - \chi(t;(t,\xi')) &= -\int_{-\infty}^{t} \widetilde{G}(t,s)\big\{f(s,x(s,t,\xi),x(\gamma(s),t,\xi)) \\
&\quad -f(s,x(s,t,\xi'),x(\gamma(s),t,\xi'))\big\}\,ds \\
&\quad -\int_{t}^{\infty} \widetilde{G}(t,s)\big\{f(s,x(s,t,\xi),x(\gamma(s),t,\xi)) \\
&\quad -f(s,x(s,t,\xi'),x(\gamma(s),t,\xi'))\big\}\,ds \\
&= -I_1 + I_2.
\end{aligned} \tag{6.1}$$

Now, we divide I_1 and I_2 as follows:

$$I_1 = \int_{-\infty}^{t-L} + \int_{t-L}^{t} = I_{11} + I_{12} \quad \text{and} \quad I_2 = \int_{t}^{t+L} + \int_{t+L}^{\infty} = I_{21} + I_{22},$$

where L is a positive constant.

By using **(A2)** combined with Proposition 3, we can see that the integrals I_{11} and I_{22} are always finite since

$$|I_{11}| \leq 2K\rho^*\mu \int_{-\infty}^{t-L} e^{-\alpha(t-s)}\, ds = \frac{2K\mu\rho^*}{\alpha} e^{-\alpha L}$$

and

$$|I_{22}| \leq 2K\rho^*\mu \int_{t+L}^{\infty} e^{-\alpha(s-t)}\, ds = \frac{2K\mu\rho^*}{\alpha} e^{-\alpha L}.$$

Now, by **(A3)** and Proposition 3, we have that

$$\begin{aligned} |I_{12}| \quad \leq \quad & \int_{t-L}^{t} K\rho^* e^{-\alpha(t-s)} \ell_1 |x(s,t,\xi) - x(s,t,\xi')|\, ds \\ & + \int_{t-L}^{t} K\rho^* e^{-\alpha(t-s)} \ell_2 |x(\gamma(s),t,\xi) - x(\gamma(s),t,\xi')|\, ds \\ \leq \quad & \int_{0}^{L} K\rho^* e^{-\alpha u} \ell_1 |x(t-u,t,\xi) - x(t-u,t,\xi')|\, ds \\ & + \int_{0}^{L} K\rho^* e^{-\alpha u} \ell_2 |x(\gamma(t-u),t,\xi) - x(\gamma(t-u),t,\xi')|\, ds. \end{aligned}$$

On the other hand, by Lemma 5.1, we have that

$$0 \leq |x(t-u,t,\xi) - x(t-u,t,\xi')| \leq |\xi - \xi'| e^{p_1 L} \quad \text{for any} \quad u \in [0,L].$$

Similarly, by using Lemmatas 2.1 and 5.1, we have that

$$0 \leq |x(\gamma(t-u),t,\xi) - x(\gamma(t-u),t,\xi')| \leq |\xi - \xi'| e^{p_1(\theta+L)} \quad \text{for any} \quad u \in [0,L].$$

The reader can deduce that the inequalities above implies

$$|I_{12}| \leq D|\xi - \xi'| \quad \text{with} \quad D = \frac{K\rho^* e^{p_1 L}}{\alpha}(1 - e^{-\alpha L})(\ell_1 + \ell_2 e^{p_1\theta}). \tag{6.2}$$

Analogously, we can deduce that

$$|I_{21}| \leq D|\xi - \xi'|. \tag{6.3}$$

For any $\varepsilon > 0$, we can choose

$$L \geq \frac{1}{\alpha} \ln \big(\frac{8K\mu\rho^*}{\alpha\varepsilon}\big),$$

which implies that $|I_{11}| + |I_{22}| < \varepsilon/2$. By using this fact combined with (6.2)–(6.3), we obtain that

$$\forall \varepsilon > 0 \;\; \exists \delta = \frac{\varepsilon}{4D} > 0 \quad \text{such that} \quad |\xi - \xi'| < \delta \Rightarrow |\chi(t;(t,\xi)) - \chi(t;(t,\xi'))| < \varepsilon$$

and the uniform continuity follows. $\square$

Lemma 6.2. *The map $\nu \mapsto L(t,\nu) = \nu + \vartheta(t;(t,\nu))$ is uniformly continuous for any t.*

Proof. We only need to prove that the map $\nu \mapsto \vartheta(t;(t,\nu))$ is uniformly continuous. In order to prove that, let ν and ν' be two initial conditions of (1.1) and define

$$\Delta = \vartheta(t;(t,\nu)) - \vartheta(t;(t,\nu')).$$

By using (4.4), we can see that Δ can be written as follows:

$$\Delta = \tag{6.4}$$
$$\begin{aligned}
&\int_{-\infty}^{t} \widetilde{G}(t,s)\big\{f(s,y(s,t,\nu)+\vartheta(s;(t,\nu)),y(\gamma(s),t,\nu)+\vartheta(\gamma(s);(t,\nu)))\\
&-f(s,y(s,t,\nu')+\vartheta(s;(t,\nu')),y(\gamma(s),t,\nu')+\vartheta(\gamma(s);(t,\nu')))\big\}\,ds+\\
&\int_{t}^{\infty} \widetilde{G}(t,s)\big\{f(s,y(s,t,\nu)+\vartheta(s;(t,\nu)),y(\gamma(s),t,\nu)+\vartheta(\gamma(s);(t,\nu)))\\
&-f(s,y(s,t,\nu')+\vartheta(s;(t,\nu')),y(\gamma(s),t,\nu')+\vartheta(\gamma(s);(t,\nu')))\big\}\,ds
\end{aligned}$$
$$= J_1 + J_2.$$

As before, we divide J_1 and J_2 as follows:

$$J_1 = \int_{-\infty}^{t-\tilde{L}} + \int_{t-\tilde{L}}^{t} = J_{11} + J_{12}, \quad J_2 = \int_{t}^{t+\tilde{L}} + \int_{t+\tilde{L}}^{\infty} = J_{21} + J_{22}.$$

By **(A2)** and Proposition 3, it is straightforward to verify that

$$|J_{11}| \leq \frac{2K\rho^*\mu}{\alpha}e^{-\alpha\tilde{L}} \quad \text{and} \quad |J_{22}| \leq \frac{2K\rho^*\mu}{\alpha}e^{-\alpha\tilde{L}}.$$

Let us define

$$||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_\infty = \sup_{s\in(-\infty,\infty)} |\vartheta(s;(t,\nu)) - \vartheta(s;(t,\nu'))|, \tag{6.5}$$

and notice that **(A3)** and Proposition 3 implies:

$$\begin{aligned}
|J_{12}| \leq\ & \frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_\infty \\
& +K\rho^*\ell_1 \int_{t-\tilde{L}}^{t} e^{-\alpha(t-s)}|y(s,t,\nu)-y(s,t,\nu')|\,ds \\
& +K\rho^*\ell_2 \int_{t-\tilde{L}}^{t} e^{-\alpha(t-s)}|y(\gamma(s),t,\nu)-y(\gamma(s),t,\nu')|\,ds \\
\leq\ & \frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_\infty \\
& +K\rho^*\ell_1 \int_{0}^{L} e^{-\alpha u}|y(t-u,t,\nu)-y(t-u,t,\nu')|\,ds \\
& +K\rho^*\ell_2 \int_{0}^{L} e^{-\alpha u}|y(\gamma(t-u),t,\nu)-y(\gamma(t-u),t,\nu')|\,ds.
\end{aligned}$$

By using Lemma 5.2, we know that

$$|y(t-u,t,\nu)-y(t-u,t,\nu')| \leq |\nu-\nu'|e^{p_2\tilde{L}} \quad \text{for any} \quad u\in[0,\tilde{L}]$$

and by using again Lemmatas 5.2 and 2.1, we have

$$|y(\gamma(t-u),t,\nu)-y(\gamma(t-u),t,\nu')|\leq|\nu-\nu'|e^{p_2(\theta+\tilde{L})}\quad\text{for any}\quad u\in[0,\tilde{L}]$$

and the reader can deduce that

$$|J_{12}|\leq\frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty+\tilde{D}|\nu-\nu'|$$

with

$$\tilde{D}=\frac{K\rho^*e^{p_2L}}{\alpha}(1-e^{-\alpha\tilde{L}})(\ell_1+\ell_2e^{p_2\theta}),$$

in addition, the following inequality can be proved in a similar way

$$|J_{21}|\leq\frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty+\tilde{D}|\nu-\nu'|.$$

By using the inequalities stated above combined with (3.1), he have

$$\begin{aligned}|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| &\leq |J_{11}|+|J_{22}|+|J_{12}|+|J_{21}|\\ &\leq \frac{4K\rho^*\mu}{\alpha}e^{-\alpha\tilde{L}}+2\tilde{D}|\nu-\nu'|\\ &\quad +\frac{2K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty,\end{aligned}$$

and we obtain

$$|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))|\quad\leq\quad\frac{4K\rho^*\mu e^{-\alpha\tilde{L}}}{\alpha(1-\Gamma^*)}+\frac{2\tilde{D}}{1-\Gamma^*}|\nu-\nu'|.$$

with Γ^* defined by

$$\Gamma^*=\frac{2K\rho^*}{\alpha}(\ell_1+\ell_2)<1.$$

Finally, for any $\varepsilon>0$, we can choose

$$\tilde{L}\geq\frac{1}{\alpha}\ln\Big(\frac{8K\mu\rho^*}{\alpha\varepsilon(1-\Gamma^*)}\Big),$$

which implies that $\frac{4K\rho^*\mu}{\alpha(1-\Gamma^*)}e^{-\alpha\tilde{L}}<\varepsilon/2$. By using this fact, we obtain that

$$\forall\varepsilon>0\;\;\exists\delta=\frac{\varepsilon}{4\tilde{D}(1-\Gamma^*)}>0\quad\text{such that}\quad|\nu-\nu'|<\delta\Rightarrow|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))|<\varepsilon$$

and the uniform continuity follows. □

6.2. **Proof of Theorem 3.** As before, we only have to prove that the maps $\xi\mapsto H(t,\xi)$ and $\nu\mapsto L(t,\nu)$ defined in the section 4 are Hölder continous.

Lemma 6.3. *For any couple ξ and ξ' such that $|\xi-\xi'|<1$, there exists $C_1>1$ such that the map $\xi\to H(t,\xi)=\xi+\chi(t;(t,\xi))$ verifies*

$$|H(t,\xi)-H(t,\xi')|\leq C_1|\xi-\xi'|^{\frac{\alpha}{p_1}}\quad\text{for any}\quad t\in\mathbb{R},$$

with p_1 defined by (5.2).

Proof. As before, we only need to prove that the map $\xi \mapsto \chi(t;(t,\xi))$ is uniformly continuous. Now, we use the the identity

$$\chi(t;(t,\xi)) - \chi(t;(t,\xi')) \quad = \quad -I_1 + I_2,$$

described by (6.1). Nevertheless, this time we consider the intervals I_1 and I_2:

$$I_1 = \int_{-\infty}^{t-T} + \int_{t-T}^{t} = I_{11} + I_{12}, \quad I_2 = \int_{t}^{t+T} + \int_{t+T}^{\infty} = I_{21} + I_{22},$$

where

$$T = \frac{1}{p_1} \ln\Big(\frac{1}{|\xi - \xi'|}\Big). \tag{6.6}$$

The reader can easily verify that

$$e^{-\alpha T} = |\xi - \xi'|^{\frac{\alpha}{p_1}} \quad \text{and} \quad e^{p_1 T} = |\xi - \xi'|^{-1}, \tag{6.7}$$

which combined with (2.11) implies that

$$|I_{11}| \leq \frac{2\mu K \rho^*}{\alpha} |\xi - \xi'|^{\frac{\alpha}{p_1}} \quad \text{and} \quad |I_{22}| \leq \frac{2\mu K \rho^*}{\alpha} |\xi - \xi'|^{\frac{\alpha}{p_1}}$$

By using **(A3)**, Proposition 3 and Lemma 5.1, we have that

$$\begin{aligned} |I_{21}| \quad &\leq \quad \int_t^{t+T} K\rho^* e^{-\alpha(s-t)} \ell_1 |x(s,t,\xi) - x(s,t,\xi')|\, ds \\ &\quad + \int_t^{t+T} K\rho^* e^{-\alpha(s-t)} \ell_2 |x(\gamma(s),t,\xi) - x(\gamma(s),t,\xi')|\, ds \\ &\leq \quad |\xi - \xi'| K\rho^* \ell_1 \int_t^{t+T} e^{(p_1 - \alpha)(s-t)}\, ds \\ &\quad + |\xi - \xi'| K\rho^* \ell_2 \int_t^{t+T} e^{-\alpha(s-t)} e^{p_1 |\gamma(s) - t|}\, ds. \end{aligned}$$

By using Lemma 2.1, we can see that

$$|I_{21}| \quad \leq \quad \big\{\ell_1 + \ell_2 e^{p_1 \theta}\big\} |\xi - \xi'| K\rho^* \int_t^{t+T} e^{(p_1 - \alpha)(s-t)}\, ds.$$

Now, by (3.4), we have that $p_1 > \alpha$. By using this fact combined with (6.7), we obtain:

$$|I_{21}| \leq \frac{K\rho^*}{p_1 - \alpha} \big\{\ell_1 + \ell_2 e^{p_1 \theta}\big\} |\xi - \xi'|^{\frac{\alpha}{p_1}}.$$

A similar estimation can be obtained for I_{12}:

$$|I_{12}| \leq \frac{K\rho^*}{p_1 - \alpha} \big\{\ell_1 + \ell_2 e^{p_1 \theta}\big\} |\xi - \xi'|^{\frac{\alpha}{p_1}}.$$

Finally, as $\alpha < p_1$ and $|\xi - \xi'| < 1$, we can conclude that

$$\begin{aligned} |H(t,\xi) - H(t,\xi')| \quad &\leq \quad |\xi - \xi'| + |\chi(t;(t,\xi)) - \chi(t;(t,\xi'))| \\ &\leq \quad \Big(1 + \frac{2K\rho^*}{p_1 - \alpha} \big\{\ell_1 + \ell_2 e^{p_1 \theta}\big\} + \frac{4\mu K \rho^*}{\alpha}\Big) |\xi - \xi|^{\frac{\alpha}{p_1}}. \end{aligned}$$

□

Lemma 6.4. *For any couple ν and ν' such that $|\nu-\nu'|<1$, there exists $D_1>1$ such that the map $\xi \to L(t,\xi)=\xi+\vartheta(t;(t,\nu))$ verifies*

$$|L(t,\nu)-L(t,\nu')| \leq D_1|\nu-\nu'|^{\frac{\alpha}{p_2}}.$$

Proof. As in the previous proof, we will start by studying the map $\nu \to \vartheta(t;(t,\nu))$. Let us recall the identity

$$|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| \quad = \quad J_1+J_2,$$

described by (6.4). As before, we divide J_1 and J_2 as follows:

$$J_1=\int_{-\infty}^{t-\tilde{T}}+\int_{t-\tilde{T}}^{t}=J_{11}+J_{12}, \quad J_2=\int_{t}^{t+\tilde{T}}+\int_{t+\tilde{T}}^{\infty}=J_{21}+J_{22},$$

with $\tilde{T}$ defined by

$$\tilde{T}=\frac{1}{p_2}\ln\Big(\frac{1}{|\nu-\nu'|}\Big).$$

The inequalities

$$|J_{11}| \leq \frac{2\mu K}{\alpha}|\nu-\nu'|^{\frac{\alpha}{p_2}} \quad \text{and} \quad |J_{22}| \leq \frac{2\mu K}{\alpha}|\nu-\nu'|^{\frac{\alpha}{p_2}}$$

can be proved analogously as before.

By using **(A3)** combined with Proposition 3 and Lemma 5.2, we can deduce that

$$\begin{aligned}
|J_{12}| \quad \leq \quad & \int_{t-\tilde{T}}^{t} K\rho^* e^{-\alpha(t-s)}\ell_1|y(s,t,\nu)-y(s,t,\nu')|\,ds \\
& + \int_{t-\tilde{T}}^{t} K\rho^* e^{-\alpha(t-s)}\ell_2|y(\gamma(s),t,\nu)-y(\gamma(s),t,\nu')|\,ds \\
& + \int_{t-\tilde{T}}^{t} K\rho^* e^{-\alpha(t-s)}\ell_1|\vartheta(s;(t,\nu))-\vartheta(s;(t,\nu'))|\,ds \\
& + \int_{t-\tilde{T}}^{t} K\rho^* e^{-\alpha(t-s)}\ell_2|\vartheta(\gamma(s);(t,\nu))-\vartheta(\gamma(s);(t,\nu'))|\,ds \\
\leq \quad & \frac{K\rho^*}{p_2-\alpha}\Big\{\ell_1+\ell_2 e^{p_2\theta}\Big\}|\nu-\nu'|^{\frac{\alpha}{p_2}}+\frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty,
\end{aligned}$$

where $p_2>\alpha$ is consequence of (3.4). Let us recall that $||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty$ is defined by (6.5).

Similarly, we can deduce that

$$|J_{21}| \quad \leq \quad \frac{K\rho^*}{p_2-\alpha}\Big\{\ell_1+\ell_2 e^{p_2\theta}\Big\}|\nu-\nu'|^{\frac{\alpha}{p_2}}+\frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||.$$

Then, we obtain

$$\begin{aligned}|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| &\leq |J_{11}|+|J_{12}|+|J_{21}|+|J_{22}| \\ &\leq \frac{2K\rho^*}{p_2-\alpha}\Big\{\ell_1+\ell_2 e^{p_2\theta}\Big\}|\nu-\nu'|^{\frac{\alpha}{p_2}}+\frac{4\mu K}{\alpha}|\nu-\nu'|^{\frac{\alpha}{p_2}} \\ &\quad +\frac{2K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty,\end{aligned}$$

which implies that

$$\begin{aligned}||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty &\leq \Big(\frac{2K\rho^*}{p_2-\alpha}\Big\{\ell_1+\ell_2 e^{p_2\theta}\Big\}+\frac{4\mu K}{\alpha}\Big)|\nu-\nu'|^{\frac{\alpha}{p_2}} \\ &\quad +\frac{2K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty.\end{aligned}$$

Now, by using (3.1), we conclude that

$$|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))|\leq \big(1-\Gamma^*\big)^{-1}\Big(\frac{2K\rho^*}{p_2-\alpha}\Big\{\ell_1+\ell_2 e^{p_2\theta}\Big\}+\frac{4\mu K}{\alpha}\Big)|\nu-\nu'|^{\frac{\alpha}{p_2}},$$

and the inequality $p_2>\alpha$ combined with $|\nu-\nu'|<1$ allows to deduce

$$\begin{aligned}|L(t,\nu)-L(t,\nu')| &\leq |\nu-\nu'|+|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| \\ &\leq \left(1+\frac{\frac{2K\rho^*}{p_2-\alpha}\big(\ell_1+\ell_2 e^{p_2\theta}\big)+\frac{4\mu K}{\alpha}}{1-\Gamma^*}\right)|\nu-\nu'|^{\frac{\alpha}{p_2}}\end{aligned}$$

and the result follows.

□

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